Time-integrated Ionospheric Correction Derivation

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1 Derivation

The measured polarization is a combination of the (unchanging) sky/astrophysical polarization, and the modulation introduced by the Faraday rotation caused by the Earth's ionosphere:

$$\tilde{P}_{\text{obs}}(\lambda^2, t) = \tilde{P}_{sky}(\lambda^2) e^{2i\lambda^2 \phi_{\text{ion}}(t)} \tag{1}$$

During imaging, the observations are integrated (averaged) over the duration of the observation (from start, t_s, to finish, t_f). Thus the measured (time-averaged) polarization contains the time-integrated effects of the ionosphere:

$$\begin{aligned}
\tilde{P}_{\text{meas}}(\lambda^2) &= \frac{1}{t_f - t_s} \int_{t_s}^{t_f} \tilde{P}_{\text{obs}}(\lambda^2, t) dt \\
&= \tilde{P}_{\text{sky}}(\lambda^2) \, \frac{1}{t_s - t_f} \int_{t_s}^{t_f} \, e^{2i\lambda^2 \phi_{\text{ion}}(t)} dt \\
&= \tilde{P}_{\text{sky}}(\lambda^2) \, \tilde{\Theta}(\lambda^2)
\end{aligned} \tag{2}$$

where $\tilde{\Theta}(\lambda^2)$ is the time-integrated effect of the ionosphere:

$$\tilde{\Theta}(\lambda^2) = \frac{1}{t_s - t_f} \int_{t_s}^{t_f} \, e^{2i\lambda^2 \phi_{\text{ion}}(t)} dt \tag{3}$$

1.1 Some notes on application/correction

In principle, since we have $\tilde{P}_{\text{meas}}(\lambda^2)$ and want $\tilde{P}_{\text{sky}}(\lambda^2)$, it's as simple as dividing by $\tilde{\Theta}(\lambda^2)$. This is slightly complicated by the presence of noise ($\tilde{n}$) in the measured data. We can comfortably ignore any time dependence in the noise, so in practice it looks more like:

$$\tilde{P}_{\text{meas}}(\lambda^2) = \tilde{P}_{\text{sky}}(\lambda^2) \, \tilde{\Theta}(\lambda^2) + \tilde{n}(\lambda^2) \tag{4}$$

$$\tilde{P}_{sky}(\lambda^2) = (\tilde{P}_{meas}(\lambda^2) - \tilde{n}(\lambda^2))/\tilde{\Theta}(\lambda^2). \tag{5}$$

From this equation we can note two effects. First is that if there is ever ionospheric conditions that cause complete depolarization ($\tilde{\Theta} = 0$), reconstruction of the original signal is impossible. This is unlikely to ever happen exactly, but since there is some uncertainty in the ionospheric RM and corresponding (unquantified) uncertainty in the time-integrated modulation, small values of $|\tilde{\Theta}|$ as likely to have large fractional uncertainties and will pass those fractional uncertainties on to the reconstructed sky polarization. Ionospheric correction when $|\tilde{\Theta}|$ is small should probably not be trusted; the threshold for where this becomes a problem has not been explored but I would guess that $|\tilde{\Theta}| \lesssim 0.3$ is likely to have large uncertainties.

Second, since $\tilde{\Theta}(\lambda^2)$ will always have an amplitude less than 1, this will have the effect of amplifying the noise by a factor of $1/|\tilde{\Theta}|$. Small values of $|\tilde{\Theta}|$ will cause, in addition to the first problem, greatly amplified noise in the data.

It should be emphasized that whenever $|\tilde{\Theta}|$ is small this is an indication of strong depolarization that can be mitigated by using a time-dependent correction. Applying a time-independent correction of the type derived here is only effective and appropriate when the time-dependent effects on the data are small enough to not significantly affect the scientific value of the data.

1.2 In Faraday depth space

An interesting addendum to this derivation is that it is possible to define an ionospheric 'RMSF'. Using $\mathcal{F}$ to denote the RM synthesis Fourier transform, we can define a function $\tilde{\Theta}(\phi)$

$$\begin{aligned} \mathcal{F}\{\tilde{P}_{meas}(\lambda^2)\} &= \mathcal{F}\{\tilde{P}_{sky}(\lambda^2)\ \tilde{\Theta}(\lambda^2)\} \\ \tilde{F}(\phi) &= \mathcal{F}\{\tilde{P}_{sky}(\lambda^2)\} * \mathcal{F}\{\tilde{\Theta}(\lambda^2)\}\} \\ &= \tilde{F}_{sky} * \tilde{\Theta}(\phi) \end{aligned} \tag{6}$$

which is the Fourier transform of $\tilde{\Theta}(\lambda^2)$.

If the approximation from the appendix (breaking $\phi_{ion}(t)$ into a sequence of linear segments) is used, each individual segment transforms into a top-hat function (or a Dirac delta function, if $\Delta\phi_j$ is zero). The proof of this is omitted for the time being.

A Semi-analytic approximation for time-averaged correction

Before I tested the stability of numeric integration of equation 3, I derived a semi-analytic approximation. This turned out to not be necessary, as the scipy numeric integral method works surprisingly well, but I am keeping the derivation here in case it ever becomes useful. I have not coded up this approximation, so I can't comment as to how it compares to using direct numeric methods.

Beginning with the definition of $\Theta(\lambda^2)$, and breaking the observation into a number of smaller time-ranges:

$$\begin{aligned}\tilde{\Theta}(\lambda^2) &= \frac{1}{t_s - t_f}\int_{t_s}^{t_f} e^{2i\lambda^2\phi_{\mathrm{ion}}(t)}dt \\ &= \frac{1}{t_s - t_f}\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} e^{2i\lambda^2\phi_{\mathrm{ion}}(t)}dt\end{aligned} \tag{7}$$

If each time range is small enough, then $\phi_{\mathrm{ion}}(t)$ can be approximated as linear, with

$$\begin{aligned}\phi_j(t) &= \phi_{\mathrm{ion}}(t_j) + \frac{\phi_{\mathrm{ion}}(t_j) - \phi_{\mathrm{ion}}(t_{j+1})}{t_{j+1} - t_j}(t - t_j) \\ &= \phi_{\mathrm{ion}}(t_j) + \frac{\Delta\phi_j}{\Delta t_j}(t - t_j)\end{aligned} \tag{8}$$

for $t_j < t < t_{j+1}$. With this approximation, the integral has an analytic solution:

$$\begin{aligned}\tilde{\Theta}(\lambda^2) &= \frac{1}{t_s - t_f}\sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} e^{2i\lambda^2(\phi_{\mathrm{ion}}(t_j) + \frac{\Delta\phi_j}{\Delta t_j}(t - t_j))}dt \\ &= \frac{1}{t_s - t_f}\sum_{j=0}^{N-1} e^{2i\lambda^2\phi_{\mathrm{ion}}(t_j)}\int_{t_j}^{t_{j+1}} e^{2i\lambda^2\frac{\Delta\phi_j}{\Delta t_j}(t - t_j)}dt \\ &= \frac{1}{t_s - t_f}\sum_{j=0}^{N-1} e^{2i\lambda^2\phi_{\mathrm{ion}}(t_j)}\int_{0}^{\Delta t_j} e^{2i\lambda^2\frac{\Delta\phi_j}{\Delta t_j}t'}dt' \\ &= \frac{1}{t_s - t_f}\sum_{j=0}^{N-1} e^{2i\lambda^2\phi_{\mathrm{ion}}(t_j)}\left[\frac{1}{2i\lambda^2\frac{\Delta\phi_j}{\Delta t_j}}\left(e^{2i\lambda^2\Delta\phi_j} - 1\right)\right]\end{aligned} \tag{9}$$

In the limit where the change in ionospheric RM is very small in a given time step, this equation is vulnerable to numerical instability (the term in round brackets goes to zero, and the fraction in square brackets goes to infinity). Taking the Taylor expansion about $\Delta\phi_j = 0$ gives a more numerically stable form:

$$
\begin{aligned}
\tilde{\Theta}(\lambda^2) &= \frac{1}{t_s - t_f} \sum_{j=0}^{N-1} e^{2i\lambda^2 \phi_{\text{ion}}(t_j)} \left[\frac{1}{2i\lambda^2 \frac{\Delta\phi_j}{\Delta t_j}} \left(\sum_{k=0}^{\infty} \frac{(2i\lambda^2 \Delta\phi_j)^k}{k!} - 1 \right) \right] \\
&= \frac{1}{t_s - t_f} \sum_{j=0}^{N-1} e^{2i\lambda^2 \phi_{\text{ion}}(t_j)} \left[\left(\frac{\Delta t_j}{2i\lambda^2 \Delta\phi_j} \sum_{k=1}^{\infty} \frac{(2i\lambda^2 \Delta\phi_j)^k}{k!} \right) \right] \\
&= \frac{1}{t_s - t_f} \sum_{j=0}^{N-1} \Delta t_j e^{2i\lambda^2 \phi_{\text{ion}}(t_j)} \left(\sum_{k=1}^{\infty} \frac{(2i\lambda^2 \Delta\phi_j)^{k-1}}{k!} \right) \\
&= \frac{1}{t_s - t_f} \sum_{j=0}^{N-1} \Delta t_j e^{2i\lambda^2 \phi_{\text{ion}}(t_j)} \left(1 + \sum_{k=2}^{\infty} \frac{(2i\lambda^2 \Delta\phi_j)^{k-1}}{k!} \right)
\end{aligned}
\tag{10}
$$

This has no closed form solution, but so long as the variation in the ionosphere in each time step is ‘Faraday-thin’ ($\lambda^2 \Delta\phi_j \ll 1$) the infinite series should quickly converge. On the other hand, this is also the regime where a direct numeric integrator will also converge. It may be that this approximation has a larger area of convergence in terms of extreme Faraday rotation, but hopefully it is never necessarily to explore this regime.